Combinatorial Networks
Week 3, Thursday

Ramsey Number for graphs

- **Definition.** For integer $k \geq 2$ and integers $s_1, s_2, ..., s_k \geq 2$, the *Ramsey number* $R_k(s_1, ..., s_k)$ is the least integer n such that any k-edge coloring of K_n has a clique K_{s_i} in color i.

- **Exercise.** $R_k(s_1, ..., s_k) < \infty$.
(1) (i) If k is even, then
$$R_k(s_1, ..., s_k) \leq R_{\frac{k}{2}}(R(s_1, s_2), ..., R(s_{k-1}, s_k));$$
(ii) If k is odd, then
$$R_k(s_1, ..., s_k) = R_{k+1}(s_1, ..., s_k, 2).$$
(2) $R_k(s_1, ..., s_k) = \sum_i R_k(s_1, ..., s_i - 1, ..., s_k)$.

- **Exercise.** $2^k \leq R_k(3, ..., 3) \leq (k+1)!$.
(Remind: For the first inequality, consider bipartite graph).

Ramsey Number for hypergraphs

- **Definition.** Let V be a finite set, $2^V = \{A \subset V\}, \binom{V}{r} = \{A \subset V : |A| = r\}, G = (V, \text{any collection of subsets of } V)$ is called a *hypergraph*, and we call $H = (V, E)$, where $E \subset \binom{V}{r}$ is a *r-uniform hypergraph*.

- **Note:** 2-uniform hypergraph = graph.

- *complete r-uniform hypergraph* is $K_n^{(r)} = (V, \binom{V}{r})$, where $|V| = n$.

- An *independent set* in r-uniform hypergraph H is a subset S of vertices containing NO hyper edge.

- A *clique* in H is a set of vertices which induced a complete r-uniform hypergraph.

- **Definition.** The *hypergraph Ramsey number* $R^{(r)}(s,t)$ is the least n such that any 2-edge-coloring of $K_n^{(r)}$ has a blue $K_s^{(r)}$ or a red $K_t^{(r)}$. The Ramsey number $R^{(r)}(s_1, ..., s_k)$ is the least n such that any k-edge-coloring of $K_n^{(r)}$ has a monochromatic clique $K_{s_i}^{(r)}$ (with color i).

- **Theorem.** For any $s, t \geq r$, the Ramsey number $R^{(r)}(s,t) < \infty$. In fact, we prove $R^{(r)}(s,t) \leq R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1)) + 1$.

- **Proof** : By induction on r, s, t.
Base case,
$$R^{(2)}(s,t) < \infty, R^{(r)}(s,r) = s < \infty, R^{(r)}(r,t) = t < \infty.$$
Inductive step: Let $n = R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1)) + 1$, consider any 2-edge-coloring of $K_n^{(r)}$ and an vertex v, $H = (V - \{v\}, \binom{V-\{v\}}{r-1})$ is a complete $(r-1)$-uniform hypergraph. Define a 2-edge-coloring on H by: $A \in E(H)$ is colored blue if and only if $A \bigcup \{v\}$ is blue

in r-uniform hypergraph $K_n^{(r)}$.
Since $n-1 = R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1))$, this 2-edge-coloring on complete $(r-1)$-uniform hypergraph H has a blue $K^{(r-1)}_{R^{(r)}(s-1,t)}$ or a red $K^{(r-1)}_{R^{(r)}(s,t-1)}$.
Case 1: There is a blue $K^{(r-1)}_{R^{(r)}(s-1,t)}$ on $V-\{v\}$, let $T = R^{(r)}(s-1,t)$. For any $A \in \binom{T}{r-1}$, $A \bigcup \{v\}$ is blue in $K_n^{(r)}$. Consider complete r-uniform hypergraph in T and the restricted 2-edge-coloring of $K_n^{(r)}$ on T. Since $T = R^{(r)}(s-1,t)$, this 2-edge-coloring has a blue $K_{s-1}^{(r)}$ or a red $K_t^{(r)}$. In the latter case, we find a red $K_t^{(r)}$ in $K_n^{(r)}$, done; In the former case, we have a blue $K_{s-1}^{(r)}$, adding v to the set, we get a blue $K_s^{(r)}$, also done.
Case 2 is similar. ∎

- **Theorem.** $R^{(r)}(s_1, ..., s_k) < \infty$.

An application of 4-uniform hypergraph on geometry

- **Definition.** In Euclidean space, a set $P \subset R^2$ is *convex*, if for any two vertices $x, y \in P$, the line segment connecting x and y are also contained in P.

- **Definition.** The *convex hull* of a set $P \subset R^2$ is the smallest convex set containing P.

- **Definition.** A set $P \subset R^2$ is in a *convex position* if NO points $x \in P$ is in the convex hull of the other points in P.

- **Definition.** A set $P \subset R^2$ is in a *general position* if NO three points are in a line.

- **Fact** 1. Among any 5 points in general position of the plane, there are always 4 points which are in convex position.

- **Proof.** Consider the convex hull of 5 points. If its convex polygon has 4 or 5 points on it, then we are done, so the convex polygon is a $\triangle abc$. Consider the other two points x, y which are inside $\triangle abc$. Consider xy-line, it must intersect two edges of $\triangle abc$, say ab and bc, then x, y, b, c are in convex position. ∎

- **Fact** 2. For any m points, if any 4 points of them are in convex position, then these m points are in convex position.

- **Proof.** Support NOT. There is a point a is contained in the convex hull of the other $m-1$ points. Consider any triangulation of $(m-1)$-polygon, there is a $\triangle ijk$ contains a, a contradiction. ∎